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THE GEOMETRY OF CHAINS ON A COMPLEX LINE*

BY JOHN WESLEY YOUNG

Introduction. This paper forms an elementary chapter in the projective geometry on a complex line, i. e., of a line whose points are isomorphic with the system of ordinary complex numbers and infinity. It furnishes a synthetic treatment of certain well-known topics in the theory of functions of a complex variable; it has close contact with recent work on the foundations of projective geometry; finally, it forms the basis for certain generalizations, which offer a powerful synthetic approach to various analytic problems. To this last aspect of the paper I shall return in detail on a future occasion.† The first two aspects I discuss briefly in this introduction.

But first, in view of the fact that the notion of a chain of points on a line does not appear to be generally familiar, it seems desirable to describe it briefly. This is most readily done in analytic language. Any point of a projective line may be conveniently represented by a single coordinate x. If the points of a line are hereby brought into one-to-one correspondence with the ordinary real numbers and infinity, we are wont to speak of a real line; if on the other hand the points of a line are hereby made isomorphic with the system of ordinary complex numbers and infinity we speak of a complex line. A *real representation* of the points of a complex line may then be obtained by the usual method for representing complex numbers by the real points of an Argand plane. The points of the line with real coordinate x form a subset $\mathfrak{C}_0$ of points on the line:— in the representation mentioned this subset $\mathfrak{C}_0$ is represented by the points of the real axis. By means of a projective transformation on the line the set $\mathfrak{C}_0$ is transformed into another or the same set $\mathfrak{C}$. Any such set of points $\mathfrak{C}$ on the line which is projective with the set $\mathfrak{C}_0$ of real points on the line is called a *chain* on the line. In the Argand representation the chains on the line evidently correspond to the circles (and straight lines) of the plane.

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† Cf., however, the last paragraph of the present paper.

The contact of this paper with the theory of functions of a complex variable is now evident. The study synthetically of the chains on a line with reference to their behavior toward the projectivities on the line and the resulting classification of the projectivities (into elliptic, hyperbolic, parabolic, and loxodromic) furnishes a synthetic derivation of well-known theorems concerning the linear fractional substitutions on a complex variable. On account of its elementary character this method of treatment may be of pedagogic value.*

The notion of a chain has been fundamental in the synthetic introduction of imaginaries into geometry since the time of von Staudt.† In the more recent work on the foundations of projective geometry it necessarily plays an important role. Pieri has indeed recently chosen the chain as one of the undefined elements in his set of assumptions for complex projective geometry.‡ More recently Professor Veblen and I § have given a set of assumptions for projective geometry in which the point and an undefined class of points called a line are the only undefined elements, and in which the chain is defined. This paper will in the sequel be referred to occasionally by the letter *A*. In its relation to the foundations the present paper forms a continuation of the one just referred to, in which it is shown how the fundamental properties of chains are derived from the assumptions laid down in *A*. As may be expected several of Pieri's assumptions occur here as theorems. From the point of view of the foundations it is interesting to note that the projective geometry of chains on a complex line is isomorphic with the metric geometry of circles in a plane.

From what has been said it is clear that the approach to the present paper may be either from the synthetic or from the analytic side. For the former reference may simply be made to *A*. The analytic description of a chain is given above. The presuppositions of the paper are given below in section 1. They are for the most part elementary notions and theorems of projective geometry; in so far as they apply to the less familiar ideas of complex projective geometry reference may be made to *A*, or they may be derived analytically without difficulty. In section 2 is discussed the system of invariant chains of an involution. Conjugate points with respect to a chain and the notion of orthogonal chains are defined and discussed in section 3. The following section brings the classification of projectivities into elliptic, hyperbolic, parabolic, and loxodromic, and the discussion of the associated systems of chains. In section 5 are then derived certain analytic criteria. Finally in section 6 the results previously obtained are used to write down a complete list of the different types of continuous groups which leave a chain invariant, i. e., of those continuous groups of linear fractional transformations which can be represented with real coefficients. This section and the paper close with a brief reference to the application of the results and methods of this paper to the synthetic treatment of the geometric theory of discontinuous groups of projectivities, i. e., of the groups of the automorphic functions of one variable.

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* If the isomorphism between the system of complex numbers and the points of a complex projective line be emphasized together with the fact that the usual Argand representation is simply a *real representation of the points of a complex line*, the beginner will no longer be troubled with the fact that the Argand plane shows only a point at infinity, whereas the projective plane has a line at infinity.

† v. Staudt, *Beiträge zur Geometrie der Lage*, Nürnberg, 1857, p. 137 ff.

‡ Pieri, Nuovi principii di geometria projettiva complessa, *Memorie della R. Accad. d. Scienze di Torino*, ser. 2, vol. 55 (1905), pp. 189–235.

§ Veblen and Young, A set of assumptions for projective geometry, *American Journal of Mathematics*, vol. 30 (1908), pp. 347–380.

SECTION 1. **Presuppositions and notation.** Points will in general be denoted by the capital letters of the alphabet such as A, B, $\cdots$ M, N. All points considered are on the same complex line. We assume the following definitions and theorems from projective geometry:

I. The *harmonic conjugate* B of a point A with respect to two others CD is a uniquely determined point, distinct from A, C, and D; this is denoted by $B = H(A, CD)$. The relation $B = H(A, CD)$ implies the relations $A = H(B, CD)$ and $C = H(D, AB)$. Each of the pairs AB, CD, is said to be *harmonic* with the other; and the two together form a *harmonic set*; this is denoted by $H(AB, CD)$.

II. 1. A *projectivity* π is a one-to-one reciprocal correspondence or transformation, whereby to any point P corresponds a unique point P'; in symbols $\pi(P) = P'$. 2. A projectivity transforms any harmonic set into a harmonic set.* 3. A projectivity is uniquely determined by three pairs of homologous points. 4. A point M for which $\pi(M) = M$ is called a double point of π; a projectivity is uniquely determined when two double points and one pair of homologous points is given. 5. If a projectivity π_1 is followed by another or the same π_2, the resultant correspondence is a projectivity denoted by $\pi_2\pi_1$; the inverse of a given projectivity π is denoted by π^{-1}. 6. Every projectivity different from the identity has at least one and not

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*On a complex line properties II 1, 2 are not sufficient to characterize a projective transformation. Cf. *A*, p. 373.

more than two double points. 7. Any two projectivities with the same double points are commutative, i.e., $\pi_2\pi_1 = \pi_1\pi_2$. 8. If π and $\pi_1\pi\pi_1^{-1}$ have the same double points we have $\pi_1\pi\pi_1^{-1} = \pi$. 9. If a projectivity π has only one double point M and we have $\pi(AA_1) = A_1A_2$, we have $H(AA_2, A_1M)$.

III. 1. A projectivity I of period two, i. e., such that $II = 1$, is called an *involution*. 2. Any involution has two distinct double points, and 3. is determined uniquely by two pairs of homologous points, in particular by its double points; the involution I determined by the two homologous pairs AA', BB' is denoted by $I(AA', BB')$. 4. If the double points of an involution I are MN, and if AA' is any homologous pair, we have $H(AA', MN)$, and conversely; a homologous pair of an involution is ususually called a conjugate pair. 5. If a projectivity interchanges two distinct points it is an involution.

IV. 1. A *chain* $\mathfrak{C} = |ABC|$ is an infinite class of points, uniquely determined by any distinct three of its points ABC. 2. Any class of points projective with a chain is a chain. 3. If PQR are any three distinct points of a chain, $S = H(P, QR)$ is a point of the chain.

V. 1. Neither double point of the involution $I(AA', BB')$ determined by a harmonic set $H(AA', BB')$ is a point of $|AA'B|$. 2. Every chain containing two conjugate pairs of an involution I but not containing the double points of I has at least one point in common with every chain through the double points. 3. Through a point P of a chain $\mathfrak{C}$ and any point Q not on $\mathfrak{C}$ there is not more than one chain that has no other point in common with $\mathfrak{C}$ than P.

The last two propositions (V, 2, 3) merit a word of comment. From the point of view of the foundations, each of them is a consequence of the other in connection with the other assumptions made in A for complex projective geometry; one of them, however, is necessary as an assumption of closure. In A we chose V, 3 for this purpose, which is there given as Assumption I 2 (A, p. 371.) The reader not interested in the foundations may simply assume these propositions as intuitionally evident from the Argand representation; chosing $H(AA', BB') = H(0\,\infty, -1\,1)$, for example, in which case the double points of I in V, 2 are i and $-i$.

SECTION 2. **The invariant chains of an involution.** Throughout this section I represents an involution whose double points are MN.

THEOREM 1. *An involution I leaves invariant every chain containing the double points M, N of I.*

Proof. Let $\mathfrak{C}$ be any chain containing the double points and let A be any point of $\mathfrak{C} = |MNA|$. Then $I(A) = A' = H(A, MN)$ (by I, 1, and III, 4). Hence A' is a point of $|MNA|$ by IV, 1, 3.

THEOREM 2. *Through every point $P(\neq M, N)$ there is an invariant chain of I, not containing MN.*

Proof. Let P' be the conjugate of P in the involution I, and let I_1 be $I(MN, PP')$. The double points QQ' of I_1 are harmonic with MN, and hence form a conjugate pair of I (III, 2, 3, 4). But we also have $H(PP', QQ')$ (III, 4). Hence the chain $|PQP'| = |P'Q'P|$ is invariant under I, and does not contain the double points MN of I (V,1).

THEOREM 3. *An invariant chain of I not through MN is cut by every chain through MN in two points harmonic with MN.*

Proof. Let $\mathfrak{C}$ be an invariant chain not through MN. Then, by V, 2, every chain through MN cuts $\mathfrak{C}$ in at least one point, which is not a double point of I. Since every chain through MN is also invariant by Theorem 1, it follows that every such chain cuts $\mathfrak{C}$ in two distinct points which are conjugates of I, and are therefore harmonic with MN (III, 4).

THEOREM 4. *If a chain $\mathfrak{C}$ not through MN meets two chains through MN in pairs of points harmonic with MN, it meets every chain through MN in points harmonic with MN.*

Proof. Let $\mathfrak{C}_1$, $\mathfrak{C}_2$ be the two chains through MN and let A_1B_1 and A_2B_2 be the pairs of points in which they meet $\mathfrak{C}$ respectively. Then MN are the double points of $I(A_1B_1, A_2B_2)$, and this involution leaves $\mathfrak{C} = |A_1B_1A_2| = |B_1A_1B_2|$ invariant. The theorem then follows from Theorem 3.

DEFINITION. A chain $\mathfrak{C}$ which cuts every chain through two given points MN in points harmonic with MN is said to be *about** MN. The two points MN are said to be *conjugate* with respect to $\mathfrak{C}$, and each is the *conjugate* or *inverse* of the other with respect to $\mathfrak{C}$; every point of a chain $\mathfrak{C}$ is said to be *conjugate with* or *inverse to* itself with respect to $\mathfrak{C}$.

THEOREM 5. *Through any point $P(\neq M, N)$ there is one and only one chain about MN.*

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* "About" is here used in a technical sense. Cf. Harkness and Morley, *Introduction to the theory of analytic functions* (London, 1898), p. 30.

Proof. By Theorems 2 and 3, there is one chain $\mathfrak{C}$ through P about MN. Moreover, using the notation in the proof of Theorem 2, we can show that $\mathfrak{C} = |PQP'|$ is the only chain through P about MN. For, if $\mathfrak{C}'$ were another chain through P about MN, $\mathfrak{C}'$ would also contain P' (fig. 1). Now, $|QMN|$ is about PP' by Theorem 4 and hence $\mathfrak{C}$ and $\mathfrak{C}'$ meet $|QMN|$ in two pairs of points QQ', RR' each of which is harmonic with PP'. But by hypothesis the pairs QQ', RR' are harmonic with MN. This is impossible, since MN and PP' cannot both be the double points of $I(QQ', RR')$ (III,4).

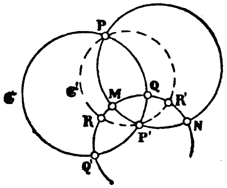

FIG. 1.

COROLLARY 1. *The totality of chains about two given points MN forms an infinite system of chains no two of which have a point in common.*

For, only one chain of the system can pass through any given point, and there is one chain of the system through every point ($\neq M, N$) of any chain through MN.

COROLLARY 2. *The chains through two points form an infinite system of chains.*

For there is one through every point on any chain about the two points.

THEOREM 6. *An involution whose double points are MN leaves invariant every chain through MN and every chain about MN, and only these.*

Proof. Clearly an invariant chain of an involution must contain both or neither of the double points (III, 4). An involution leaves invariant every chain through the double points by Theorem 1. Every chain about MN meets every chain through MN in a pair of points harmonic with MN. Hence, any point of a chain about MN is conjugate with a point of the same chain (III, 4). Further, every invariant chain not through MN must meet every chain through MN in points harmonic with MN (Theorem 3), and is therefore about MN (III, 4).

COROLLARY. *Through every point* $P(\neq M, N)$ *pass two and only two invariant chains of the involution whose double points are* MN.

It should be here noted that any two distinct points MN define two infinite systems of chains, viz., the system through MN and the system about MN, such that through every point distinct from M and N passes one and only one chain of each system and such that every chain of one system meets each chain of the other in points harmonic with MN. These two systems are of great importance in the geometry of chains on a line.

THEOREM 7. *If a projectivity* π *with distinct double points* MN *leaves a chain through* MN *invariant, it leaves every chain through* MN *invariant.*

Proof. Let $\mathfrak{C}$ be the given invariant chain through MN, and let $A(\neq M, N)$ be any point of $\mathfrak{C}$, and $B(\neq M, N)$ any point of any other chain $\mathfrak{C}'$ through MN. Then if π_1 is determined by $\pi_1(MNA) = MNB$, we have $\pi_1(\mathfrak{C}) = \mathfrak{C}'$. But $\pi_1\pi\pi_1^{-1}$ evidently leaves $\mathfrak{C}'$ invariant, and $\pi_1\pi\pi_1^{-1} = \pi$, since π, π_1 have the same double points (II, 7). Hence, π leaves $\mathfrak{C}'$ invariant.

SECTION 3. Conjugate points and orthogonal chains.

THEOREM 8. *If a chain,* $\mathfrak{C}$, *is about two points* MN, *and* π *is any projectivity, then* $\pi(\mathfrak{C})$ *is about* $\pi(M)\,\pi(N)$.

Proof. If any chain, $\mathfrak{C}_1$, through $\pi(M)\,\pi(N)$ failed to cut $\pi(\mathfrak{C})$ in points harmonic with $\pi(M)\,\pi(N)$, $\pi^{-1}(\mathfrak{C}_1)$ would be a chain through MN not cutting $\mathfrak{C}$ in points harmonic with MN.

COROLLARY 1. *If* MN *are conjugate with respect to* $\mathfrak{C}$, $\pi(M)\,\pi(N)$ *are conjugate with respect to* $\pi(\mathfrak{C})$.

COROLLARY 2. *Every projectivity with distinct double points* MN *leaves invariant each of the systems of chains through* MN *and about* MN.

DEFINITION. Two chains are said to *meet orthogonally* or to be *orthogonal*, if one contains two points which are conjugate with respect to the other.

THEOREM 9. *Every projectivity transforms a pair of orthogonal chains into a pair that is orthogonal.*

This is an immediate consequence of Theorem 8.

THEOREM 10. *If* $\mathfrak{C}$ *and* $\mathfrak{C}'$ *are two chains meeting orthogonally in* PQ, *then every pair of points* MN *on* $\mathfrak{C}(\mathfrak{C}')$ *which are harmonic with* PQ *are conjugate with respect to* $\mathfrak{C}'$ $(\mathfrak{C})$.

Proof. Let AB on $\mathfrak{C}$ be conjugate with respect to $\mathfrak{C}'$, so that we have $H(AB, PQ)$ (fig. 2). Let $A'B'$ be the double points of $I(AB, PQ)$. $A'B'$ are on $\mathfrak{C}'$. For $I(PQ, A'B')$ leaves $|A'PB'|$ invariant and this chain must therefore be about AB; but there is only one chain through P and about AB. Then $\mathfrak{C}$ cuts $\mathfrak{C}'$ and $|AA'B|$ in pairs of points harmonic with $A'B'$. Hence $A'B'$ are conjugate with respect to $\mathfrak{C}$: i. e., *each of two orthogonal chains contains a pair of points conjugate with respect to the other.* Now, as before, let AB be conjugate with respect to $\mathfrak{C}'$, and let $\pi(PQAB) = PQMN$, which is possible since AB and MN are each harmonic with PQ. π transforms $\mathfrak{C}$ into itself since $|PQA| = |PQM|$ and therefore also leaves $\mathfrak{C}'$ invariant (Theorem 7). Hence, MN are conjugate with respect to $\mathfrak{C}'$ (Theorem 8, Corollary 1).

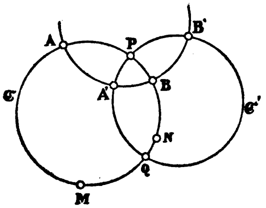

FIG. 2.

COROLLARY. *Through any point of either of two orthogonal chains which is not on both passes a chain orthogonal to both.*

This is the chain through the given point and about the points common to the two given chains.

THEOREM 11. *The inverse of any point* P *with respect to a given chain* $\mathfrak{C}$ *is a uniquely determined point.*

Proof. If P is on $\mathfrak{C}$ the theorem is immediate from the definition. If P is not on $\mathfrak{C}$, let AB be any pair of points conjugate with respect to $\mathfrak{C}$.*

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* Such a pair exists; e. g., the double points of $I(CC', DD')$, where $H(CC', DD')$ is any harmonic set on $\mathfrak{C}$.

The chain $|ABP|$, orthogonal to $\mathfrak{C}$, meets $\mathfrak{C}$ in two points MN. Hence $Q = H(P, MN)$ is the inverse of P with respect to $\mathfrak{C}$ (Theorem 10). This point Q is unique, for if there were another Q', $|PQQ'|$ would cut $\mathfrak{C}'$ in two points harmonic with PQ and with PQ' which is absurd (I, 1).

THEOREM 12. *Through any two distinct points AB there is one and only one chain orthogonal to a given chain $\mathfrak{C}$.*

Proof. Suppose A is not on $\mathfrak{C}$, and let A' be the inverse of A with respect to $\mathfrak{C}$. Then $|AA'B|$ is orthogonal to $\mathfrak{C}$ and it is the only chain through AB with this property, since any such chain must contain A'. This applies whether B is on $\mathfrak{C}$ or not. If both A and B are on $\mathfrak{C}$, let MN be any pair on $\mathfrak{C}$ harmonic with AB. Then the chain through A about MN contains B. This chain then satisfies the conditions of the theorem. It is the only one, since every such chain must be through A and about MN and there is only one such (Theorem 5).

SECTION 4. Hyperbolic, elliptic and parabolic projectivities.

THEOREM 13. *If a non-involutoric projectivity, π, leaves invariant a chain $\mathfrak{C}$, then π leaves invariant one and only one chain through every point which does not coincide with a double point of π.*

Proof. 1) Let π have two distinct double points MN (II, 6). Let A $(\neq M, N)$ be any point of $\mathfrak{C}$, and B any other point distinct from MN. Let π_1 be determined by $\pi_1(MNA) = MNB$. $\pi(\mathfrak{C}) = \mathfrak{C}'$ is then a chain through B. Since $\pi_1\pi\pi_1{}^{-1}$ clearly leaves $\mathfrak{C}'$ invariant and $\pi_1\pi\pi_1{}^{-1} = \pi$ (II, 7), it follows that π leaves $\mathfrak{C}'$ invariant. Moreover, suppose $\mathfrak{C}'_1$, were another chain through B which is left invariant by π. $\mathfrak{C}'_1$ has another point B' in common with $\mathfrak{C}'$, since B is not a double point. π must then simply interchange BB', in which case π would be involutoric (III, 5), contrary to hypothesis. Hence $\mathfrak{C}'$ is the only invariant chain through B.

2) Let π have only one double point M (II, 6). Let $A (\neq M)$ be any point of $\mathfrak{C}$, and B $(\neq M)$ any other point. Let $\pi_1(MMA) = MMB$; the theorem then follows as before.

COROLLARY 1. *If a projectivity leaves two distinct chains through a point distinct from a double point invariant, it is an involution.*

COROLLARY 2. *The system of invariant chains of a non-involutoric projectivity with distinct double points consists of all the chains* $^{\textit{through}}_{\textit{about}}$ *the double points, if one invariant chain is* $^{\textit{through}}_{\textit{about}}$ *them.*

For, if $\mathfrak{C}$ is $\genfrac{}{}{0pt}{}{\text{through}}{\text{about}}$ MN, the projectivity π_1 transforms $\mathfrak{C}$ into another chain $\genfrac{}{}{0pt}{}{\text{through}}{\text{about}}$ MN (Theorem 8); and every chain $\genfrac{}{}{0pt}{}{\text{through}}{\text{about}}$ MN can be obtained in this way.

Definition. A projectivity with two distinct double points which leaves a chain invariant, is called *hyperbolic* or *elliptic* according as an invariant chain does or does not contain the double points*. A projectivity with two double points which does not leave any chain invariant is called *loxodromic*. A projectivity with only one double point is called *parabolic*.

This classification is mutually exclusive, except that an involution is both elliptic and hyperbolic. Moreover, *every projectivity belongs to one of these four classes*. To show this we need only show that, *if a projectivity* π, *with two double points leaves invariant a chain containing one double point this chain also contains the other.*

To prove this let MN be the double points of π and let $\mathfrak{C}$ be an invariant chain of π containing M. Let P ($\neq M$, N) be any other point of $\mathfrak{C}$ and let I be the involution in which MN are conjugate and in which P is a double point. Then we have $I\pi I = \pi$, since $I\pi I$ has the same double points as π (II, 8). I transforms $\mathfrak{C}$ into a chain $\mathfrak{C}'$ through N, which is invariant under $I\pi I = \pi$. $\mathfrak{C}$ and $\mathfrak{C}'$ are both invariant chains of π and both contain P. Hence either $\mathfrak{C} = \mathfrak{C}'$, or π is an involution (Theorem 13).

Theorem 14. *Every hyperbolic projectivity leaves every chain through the double points invariant and, unless it is involutoric, only these. Every elliptic projectivity leaves every chain about the double points invariant and, unless it is involutoric, only these.*

Proof. The first part of this theorem is contained in Theorem 13, Corollary 2. The second part will likewise follow from this corollary, if we show that every elliptic projectivity, π, must leave one chain about the double points invariant. Let $\mathfrak{C}$ be invariant under π; it does not then contain either of the double points MN. We note first that there exists a chain, $\mathfrak{C}'$, through the double points MN meeting $\mathfrak{C}$ in two distinct points. For let M' be the inverse of N with respect to $\mathfrak{C}$. If we have $M' = M$, our theorem is proved. Suppose $M' \neq M$; the chain $|M'MN|$ meets $\mathfrak{C}$ in two points AB. Now, let $\pi(AA_1BB_1) = A_1A_2B_1B_2$; A_1, A_2, B_1, B_2 are then on $\mathfrak{C}$. Further let π_1

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* This definition of elliptic and hyperbolic projectivities seems the most natural and agrees with the definition given by some writers; others make the class of hyperbolic substitutions less extensive. Cf. footnote page 46.

be determined by $\pi_1(MNA) = MNB$. Then we have $\pi_1^{-1}(B_1) = \pi_1^{-1}\pi\pi_1(A) = \pi(A) = A_1$, since π, π_1 are commutative (II, 7). Hence $\pi_1(A_1) = B_1$. Similarly, we obtain $\pi_1(AA_1A_2) = BB_1B_2$. Hence π_1 leaves $\mathfrak{C}$ invariant. Also, π_1 leaves $\mathfrak{C}' = |MNA| = |MNB|$ invariant. Hence, π_1 is an involution (Theorem 13, Corollary 1), and therefore $\mathfrak{C}$ is about MN. (Theorem 6).

COROLLARY. *Conversely, to every system of chains* ${}^{\textit{through}}_{\textit{about}}$ *two points* MN *corresponds an infinite set of* ${}^{\textit{hyperbolic}}_{\textit{elliptic}}$ *projectivities, each of which leaves invariant every chain* ${}^{\textit{through}}_{\textit{about}}$ MN, *and permutes among themselves the chains* ${}^{\textit{about}}_{\textit{through}}$ MN.

For, if AB are any two points of a chain through MN, the projectivity π determined by $\pi(MNA) = MNB$, leaves each chain of the system through MN invariant; and if AB are any two points of a chain about MN the projectivity determined in the same way will leave every chain of the system about MN invariant.

Loxodromic projectivities, moreover, exist. For, if in the above AB are two points not on the same chain through MN nor on the same chain about MN, the projectivity there described will necessarily be loxodromic.

THEOREM 15. *Every parabolic projectivity* π *with double point* M *leaves invariant each chain of a system, any two of which have* M *and no other point in common. Through any point* $A(\neq M)$ *passes one and only one chain of this system.*

Proof. Let $\pi\ (AA_1) = A_1A_2$; then we have $H(AA_2, A_1M)$ (II, 9). π therefore leaves $|MAA_1| = |MA_1A_2|$ invariant. This also shows that every invariant chain contains M. Finally, two invariant chains cannot have any point other than M in common, since such a point would have to be a double point (or by Theorem 13).

DEFINITION. Two chains with a point M and no other point in common are said to be *tangent* or to *touch at* M; the point M is called the *point of tangency* or *of contact.*

We have just seen that the invariant chains of a parabolic projectivity with double point M form a system of chains mutually tangent at M.

THEOREM 16. *Any chain* $\mathfrak{C}$ *through the double point* M *of a parabolic projectivity,* π, *is transformed by* π *into another tangent to* $\mathfrak{C}$ *at* M *unless* $\mathfrak{C}$ *is an invariant chain of* π.

Proof. Suppose $\pi(\mathfrak{C}) = \mathfrak{C}'$ has another point in common with $\mathfrak{C}$; let it be denoted by A_1. Let A_0 and A_2 be determined by $\pi(A_0 A_1) = A_1 A_2$. Then clearly $\mathfrak{C} = |A_0 A_1 M|$ and $\mathfrak{C}' = |A_1 A_2 M|$. But we have $H(A_0 A_2, A_1 M)$, whence would follow $\mathfrak{C} = \mathfrak{C}'$.

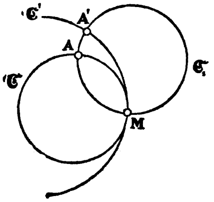

FIG. 3.

THEOREM 17. *If two chains $\mathfrak{C}$ and $\mathfrak{C}_1$ meet orthogonally in a point M, every chain tangent to $\mathfrak{C}$ at M meets every chain tangent to $\mathfrak{C}_1$ at M orthogonally.*

Proof. Let A be the second point of intersection of $\mathfrak{C}, \mathfrak{C}_1$ (fig. 3). Let $\mathfrak{C}'$ be any chain tangent to $\mathfrak{C}$ at M. $\mathfrak{C}'$ will meet $\mathfrak{C}_1$ in a point A' distinct from M; for, if M were the only point common to $\mathfrak{C}'$, $\mathfrak{C}_1$, we should have two chains, $\mathfrak{C}$ and $\mathfrak{C}_1$, through A tangent to $\mathfrak{C}'$ at M (V, 3). The parabolic projectivity determined by $\pi(MMA) = MMA'$ leaves $\mathfrak{C}_1$ invariant and transforms $\mathfrak{C}$ into $\mathfrak{C}'$. Hence $\mathfrak{C}'$ and $\mathfrak{C}_1$ are orthogonal (Theorem 9); i.e., any chain tangent to $\mathfrak{C}$ at M meets $\mathfrak{C}_1$ orthogonally. That it meets any chain tangent to $\mathfrak{C}_1$ at M orthogonally follows by reasoning entirely similar to that just given.

COROLLARY. *Given a system S of chains mutually tangent at a point M, there exists a second system S_1 of chains mutually tangent at M such that every chain of S meets every chain of S_1 orthogonally.*

SECTION 5. **Analytic formulation.** The analytic criteria for the above classification of projectivities are now readily obtainable. If, in connection for example with the developments of A, Section 2 ff., we choose any three points as 01∞, the points of the line are made isomorphic with the system of ordinary complex numbers and ∞. The real numbers then correspond to the chain $|0\,1\,\infty|$; the purely imaginary numbers to the chain $|0 i \infty|$, which is the chain through $0\,\infty$ orthogonal to $|0\,1\,\infty|$.

Definition. The *absolute value* of a number x is the positive number in which the chain through x and about 0∞ meet $|01\infty|$. Two numbers then have the same absolute value, if they are on the same chain about 0∞.

Two numbers are said to have the same *amplitude*, if they are on the same chain through $0\,\infty$.

With this analytic representation the projectivities are given by the linear fractional transformations

$$\pi(x) = x' = \frac{ax + b}{cx + d}\,, \quad ad - bc \neq 0,$$

where x, x', a, b, c, d are complex numbers. If a projectivity, π, has two distinct double points x_1, x_2, and if x, x' is any pair of homologous points of π, it is a well-known theorem of projective geometry, that the double ratio

$$(1) \qquad \frac{x - x_2}{x - x_1} \cdot \frac{x' - x_1}{x' - x_2} = k$$

is constant for all pairs $x\,x'$. The projectivity π may therefore in this case be written

$$(2) \qquad \frac{x' - x_1}{x' - x_2} = k \cdot \frac{x - x_1}{x - x_2}.$$

The desired criteria are then obtained as follows:

π is parabolic, if we have $(d - a)^2 + 4bc = 0$.

The double points of π will be distinct, if we have $(d - a)^2 + 4bc \neq 0$.

If in (1) we regard x_1, x_2, x as fixed and x', k as variable, this defines a projectivity from x' to k, whereby x_1, x_2, x are transformed respectively into $0\;\infty\;1$. By this transformation the chain through x and $x_1\,x_2$ is transformed into $|0\;1\;\infty|$; whereas the chain through x and about x_1x_2 is transformed into the chain through 1 and about $0\,\infty$. It follows, then, that $x\,x'$ are on the same chain through x_1x_2, if and only if k is real; and that $x\,x'$ are on the same chain about $x_1\,x_2$, if and only if the absolute value of k is 1. This gives the result desired:

A projectivity (2) *is hyperbolic, if k is real;** *elliptic, if the absolute value of k is* 1; *loxodromic, for any other values of k. An involution is characterized by the value $k = -1$*, which follows directly from the fact that an involution is both elliptic and hyperbolic, and the only real number different from 1 ($k = 1$ gives the identical projectivity) whose absolute value is 1 is -1; or from (1) and III, 4.

Since the necessary and sufficient condition that a projectivity $x' = (ax + b)/(cx + d)$ have real coefficients only is that it leave the chain $|0\ 1\ \infty|$ invariant, it follows readily that a projectivity can be transformed into one with real coefficients, if and only if it leaves a chain invariant. For, any projectivity transforming an invariant chain into $|0\ 1\ \infty|$ will transform the given projectivity into one leaving $|0\ 1\ \infty|$ invariant. This result may be stated as follows:

Every hyperbolic, elliptic, or parabolic projectivity can be represented analytically by a linear fractional transformation with real coefficients; and conversely, a linear fractional transformation with real coefficients is either hyperbolic, elliptic, or parabolic.

SECTION 6. **Groups of projectivities with invariant chain.**

In view of the simplicity of the reasoning it seems worth while to apply the results obtained above to the enumeration of the distinct types of continuous groups of projectivities which leave a chain invariant; or in other words, which may be represented analytically with real coefficients.

Lie's enumeration makes no distinction between real and complex; his parameters mean complex parameters. From this point of view there are four distinct types of continuous groups of projectivities, viz:

G_3: The set of all projectivities on a line.

G_2: The set of all projectivities leaving a given point invariant.

G_1: The set of all projectivities leaving two distinct points invariant.

G_1': The set of all parabolic projectivities leaving a given point invariant.

Every continuous group of projectivities in which the parameters are complex can be transformed into one and only one of these types.* In seeking the different types of groups leaving a given chain $\mathfrak{C}$ invariant, we naturally regard two groups as belonging to the same type, if and only if one can be transformed into the other by a projectivity *leaving* $\mathfrak{C}$ *invariant.* Our previous results enable us to write down at once the following five distinct types:

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* This definition of hyperbolic projectivities agrees e. g., with the definition of Enriques, *Vorlesungen über projective Geometrie* (Leipzig, 1903), p. 101; Goursat, *Cours d'analyse*, vol. 2 (Paris, 1905), p. 73; Burkhardt, *Theorie der analytischen Funktionen* (Leipzig, 1897), p. 38; with many authors a projectivity is hyperbolic only when k is real and *positive*: Osgood, *Funktionentheorie* (Leipzig, 1907), p. 225, and apparently the majority of books on the theory of functions.

RG_3 : The set of all (hyperbolic, elliptic, and parabolic) projectivities leaving $\mathfrak{C}$ invariant.

RG_2 : The set of all (hyperbolic and parabolic) projectivities leaving $\mathfrak{C}$ and a given point of $\mathfrak{C}$ invariant.

RGh_1: The set of all (hyperbolic) projectivities leaving $\mathfrak{C}$ and two points of $\mathfrak{C}$ invariant.

RGe_1: The set of all (elliptic) projectivities leaving $\mathfrak{C}$ and two points conjugate with respect to $\mathfrak{C}$ invariant.

RGp_1: The set of all parabolic projectivities leaving $\mathfrak{C}$ and a given point of $\mathfrak{C}$ invariant.

That there can be no other types, follows readily from Lie's theorem † that if in a continuous group with real parameters the parameters are allowed to take on all complex values, there results a continuous group with complex parameters provided the transformations of the group are expressed by analytic functions.

Attention may also be called to the evident possibility of deriving synthetically much of the geometric theory of *discontinuous* groups of linear fractional substitutions on a complex variable ‡ by the methods exhibited in this paper. While probably no great simplification would arise from this method of treatment in the case of one variable, it is a significant fact that the corresponding analysis for linear fractional substitutions on two complex variables is very involved, while the geometry of two dimensional chains in a plane (which forms the subject of a paper to be published in the near future) is comparatively very simple. It seems likely therefore that the synthetic approach to the geometric theory of the groups of the automorphic functions

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[*] Lie, *Vorlesungen über continuierliche Gruppen* (Leipzig, 1893), pp. 115 ff.

[†] Lie, *Theorie der Transformationsgruppen*, vol. 3, Leipzig, 1893, p. 362.

[‡] Cf., for example, the classical paper by Poincaré, Sur les groupes fuchsiens, *Acta Mathematica*, vol. 1, (1881), p. 1.

of two independent variables will yield results of value. For any such approach the methods and results of the present paper will form a starting point. Finally, we would note the connection of the present paper with the inversion geometry in the plane. A further synthetic study of the transformation which consists in replacing each point by its inverse with respect to a chain gives a natural approach to the geometry of inversion and may well form the starting point of an investigation looking toward the setting up of a set of assumptions characterizing this geometry. The solution of this problem is greatly to be desired.*

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*Cf. Kasner, The Present Problems of Geometry, *Bulletin of the American Mathematical Society*, vol. 11 (1905), p. 290.